Vertical and horizontal components of the electric background field at the sea bottom

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**ABSTRACT**

The natural E-field variations measured at the sea bottom, and the magnitude of the different field components compared in the light of the theory for induction caused by ocean surface waves. At shallow sea depths of 107–122 meters only the vertical component carries an observable effect of ocean waves, whereas the horizontal field is dominated by the larger magnetotelluric noise. This agrees well with theoretical predictions.

**INTRODUCTION**

Understanding the electromagnetic noise that reaches the bottom of the sea is of fundamental interest in a range of contexts. In magnetotellurics, the natural electromagnetic field variations is used to detect resistive or conductive bodies in the subsurface. Marine controlled source electromagnetic measurements (CSEM) rely on the resolution of a signal that must be stronger than the background noise. In many circumstances, the averaging techniques applied to reduce the noise in the signal have problems in the frequency range of the ocean waves. It is, therefore, of both fundamental and technological interest to understand the effect of ocean waves. The effects of the air-sea boundary give qualitatively different noise spectra in the different E-field components, and it is the purpose of this paper to compare the two.

In general, the sea-bottom EM background signal is generated by a large variety of mechanisms; see, for instance, Filloux (1973) or Larsen (1973). These mechanisms may be external or internal to the receivers, and they range from the low frequency ionospheric (MT) signal via the hydrodynamically generated noise in intermediate frequencies to the high-frequency Nyquist noise that comes from the electronics of the instruments involved.

The sea-bottom measurements, which are obtained by the recently developed equipment of PetroMarker, include the vertical-field component, which contains the main effect of ocean waves. The effect, however, is too small to be observable in strictly horizontal components. The reasons are that, in the absence of strong 3D heterogeneities, the wave-induced effects are much weaker and the overall magnetotelluric noise is much larger in the horizontal components. Only recently has it been possible to measure the relatively weak vertical component with appreciable accuracy in the surface wave range of frequencies. The challenge is that even minute deviations from the vertical antenna orientation will cause contamination of the signal and that the motion of conventional receivers has generally induced a large noise level. Our measurements are accurate in terms of signal-to-noise ratio, resolving field strengths down to 0.5 nV/m, and in terms of the verticality of the receiver antenna, because it is aligned with the direction of gravity to within 0.1°. This allows for a separation between horizontal and truly vertical-field components, which, to our knowledge, is unprecedented.

E-field variations due to ocean waves have been studied theoretically by many authors (Longuett-Higgins, 1950; Longuett-Higgins et al., 1954; Weaver, 1965; Sanford, 1971; Podney, 1975; Cox et al., 1978; Petersen and Poehls, 1982; Chave, 1984; Davey and Barwes, 1985; Ochadlik, 1989). Podney (1975), in particular, has provided a general 3D formalism for calculating the magnetic field due to ocean surface waves over a layered earth, whereas Chave (1983) treats the 2D case of finite sea depth, which applies most directly to our present measurements.

Earlier measurements of the vertical electric field component have been based on cable receivers with a mobile buoy supporting the top of the receiver cable, as in the works of Bindoff et al. (1986), Chave and Filloux (1985), and Harvey (1984). Such receivers are highly susceptible to the kind of motion induced noise that is caused by sea bottom currents. However, because these studies were focused on the frequency around and below one cycle per day, they
could not pick up this noise, which is in the range of about 0.1 Hz. This study, on the other hand, relies crucially on receiver technology, which is improved to screen the antenna structure from all hydrodynamic force-fluctuations.

Our measurements are in a relatively high frequency range compared to earlier studies. A broad field of both theoretical treatments and measurements of the low frequency, horizontal-field component exists, as in Luther et al. (1991), Chave and Luther (1990), and Nilsson et al. (2007), where the focus is mainly on the effects of low-frequency sea currents.

Also, our results strongly indicate that the dominant motion induced signal is that of the surface waves and not the independent signal that is generated by vorticity (turbulence) around the receiver structure (Sanford et al., 1999).

As a result, the theoretical prediction of the sea bottom vertical E-field component is well confirmed by our measurements in the ocean wave range of frequencies and at the shallow depths. At the greater depth the ocean-wave effect is screened out.

**MEASUREMENT TECHNIQUE**

The measurements are carried out by 10-meter high tripods positioned on the sea bottom from a ship at water depths of \(d = 106 - 122\ m\) and \(d = 313\ m\) (Holten et al., 2009). The vertical tripod antenna is suspended like a pendulum with lead, lead-chloride electrodes at the extreme of this pendulum arm. Figure 1 shows the screened pendulum, with its fabric coating, as it is ready to be placed in the sea. The screening, which has a tapered cylindrical form of roughly 1-meter bottom diameter, shields the antenna from hydrodynamic forces and is a key factor in ensuring both accuracy and verticality. The horizontal antennas are located in the base of the structure, and the three sets of horizontal electrodes are seen as white cylinders on the base poles. The vertical and horizontal E-field components are measured in periods of 45 minutes, and their power spectra are calculated. The sampling rate is 1000 Hz. Through independent GPS measurements, the heave-motion of the ship is recorded and the corresponding power-spectrum is calculated. These heave measurements are challenging, because the GPS must resolve displacements, which are in the range less than the amplitude of the ship’s vertical movement. This amplitude may be as low as a few centimeters for the lowest frequencies. The eigenfrequency \(f\) of the boat’s heave-motion may be estimated by the boat’s weight to be roughly \(\omega = 2\pi f = 2\ Hz\), which lies above the relevant wave frequencies. So, although this frequency is observable in the heave-spectrum, it lies above the wave frequencies and in a frequency range that does not affect the sea-bottom E-fields.

**POWER-SPECTRUM IN THE DOMAIN OF THE OCEAN WAVE FREQUENCIES**

Magnetotelluric noise, which comes from electromagnetic activity in the ionosphere, lighting, and tornadoes around the earth, appears largely in the horizontal E-field components. This follows directly from Maxwell’s equations and the large scale-separation between the thickness and the horizontal dimensions of the atmosphere. However, the noise induced by ocean waves enters mainly in the vertical-field component, when measurements are made on the sea bottom, which is generally the case in CSEM surveys.

The reason for this is the combination of electric and hydrodynamic boundary conditions. There can be no electric current through the sea surface, and there is no water flow through the sea bottom. When conductive sea water is set in motion by the surface waves, the Lorentz force created by the water velocity and the earth’s magnetic background field, creates time-dependent electric currents. Because of induction, these currents spread out diffusively with a diffusivity \(D = 1/(\sigma\mu_0)\), where \(\sigma\) is the sea water conductivity and \(\mu_0\) is the magnetic permeability (Weaver, 1965). It is instructive to compare the characteristic diffusion time \(1/(D\lambda^2)\) to the wave period \(2\pi/\omega\), where \(\omega\) and \(\lambda\) are the angular frequency and wavenumber of the ocean waves. The wavenumber \(k\) is related to the frequency through the shallow water dispersion relation, as given by Landau and Lifshitz (1959):

\[
\omega^2 = gk \tanh(kd),
\]

where \(g\) is the acceleration of gravity. The ratio of the diffusion time to the wave period,

\[
\beta = \frac{\mu_0\sigma\omega}{2k^2},
\]

is generally small, as may be seen in Figure 3. At least, this is the case for the frequency range \(\omega > 0.1\) of primary interest relative to the ocean waves. This means that the inductive currents come close

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**Figure 1.** Tripod antenna structure. The pivot point P is shown, as is the positions of the vertical antenna electrodes, Ez, and the horizontal antenna electrodes, Ex.

**Figure 2.** Sketch of the wave of wavelength \(\lambda\) in a sea of depth \(d\) with coordinate system. The positive y-direction is parallel to the geomagnetic north and points into the plane.
to their steady state values where diffusion has had the time to relax. This observation allows us to estimate the electric fields by ignoring external induction effects and only looking at the ocean motional induction component in the presence of a stationary magnetic field.

Assuming that the sea water is neutral, so that the charge density vanishes, Ohm’s law gives the following current density in the seawater:

$$j = \sigma (E + \mu_0 u \times H),$$

where $E$ is the electric field, $u$ the water velocity, $H$ the magnetic field, and all fields are measured relative to the sea-bottom frame of reference. In a frame moving with the seawater the electric field would be measured as $E + \mu_0 u \times H$ and the electric current density would be the same, $j$.

Figure 2 shows the coordinate system with the $y$-pointing along the direction of the geomagnetic north. Now, assuming a layered earth model where there are no horizontal variations in charge density, a stationary current will flow freely in all horizontal directions and there can be no horizontal electric field.

On the other hand, there may well be charge density variations in the vertical direction, both on the surface and the bottom of the sea. These charge densities will in particular cause the sea surface boundary condition $j_z = 0$ to hold. It is shown in Appendix A that, due to lack of source terms for the $j_z$-field, surface waves over a layered earth model will give $j_z = 0$ everywhere. In particular, this means that

$$E_z = -\mu_0 (u \times H)_z = u_x B_y,$$

at the sea bottom. Here we have taken the $y$-axis along the magnetic north direction. For this reason any velocity component in the $y$-direction will not contribute to $E_z$.

In general a surface wave will be moving in some angle $\Theta$ to the magnetic north. However, we shall start by considering a wave moving in the east-west direction for which $\Theta = \pi/2$, i.e., we consider a plane ocean wave of frequency $\omega$ with a horizontal wave vector $k$ in the $x$-direction, so that $u \propto e^{i (\omega t + kx)}$. An inviscid and incompressible velocity field from waves of small amplitude compared to their wavelength will satisfy the Laplace equation. In particular this is true for the vertical component of the field $\nabla^2 u_z = 0$. If we add the boundary condition of the surface velocity

Figure 3. The dispersion relation $k = k(\omega)$ (red curves) and the parameter $\beta = \beta(\omega)$ (green curves), which estimates the relative magnitude of the diffusion time to the wave period. The dashed curves show the $d = 300$ m results, and the full curves the $d = 122$ m results.

Figure 4. Shallow sea ($d = 107$ m) powerspectra $|E_z(f)|^2$, $|E_x(f)|^2$ and their ratio, $|E_z(f)/E_x(f)|^2$, taken at different times.
at $z = 0$, and $u_z(d) = 0$ at the bottom, it may be shown (Landau and Lifshitz, 1959) that

$$u_z(x, t) = -h_0 \omega \frac{\sin(k(z - d))}{\sinh(kd)} e^{i(\omega t + kx)},$$

where $k = |k|$ and $h_0$ is the wave height. The incompressibility condition $\nabla \cdot \mathbf{u} = 0$ gives the other nonzero velocity component $u_x = -\partial_x u_z/(ik)$, where $\partial_x$ denotes the derivative with respect to $x$, so that

$$u(x, t) = -h_0 \omega \frac{\sinh(kd)}{\sinh(kd)} (e_x i \cosh(k(z - d))) + e_z \sinh(k(z - d)) e^{i(\omega t + kx)}.$$

Using this velocity field we can calculate $E_z$ in response to the wave of height $h_0$ and frequency $\omega$. However, our theory is entirely linear, so it will apply to a superposition of waves of different frequencies as well. In that case we need only to consider each separate frequency, and this is simply done by allowing $h_0$ to depend on frequency. Moreover, if we make the approximation that all frequency components of the wave move in the same direction, we can account for the full effect of the wave propagating in the $\Theta$-direction by the replacement $h_0 \rightarrow h(\omega) \sin(\Theta)$. The power-spectrum of the electric field is then given as

$$|E(d, \omega)|^2 \approx |E_z(d, \omega)|^2 = \frac{|h(\omega)|^2 \sin^2(\Theta) \omega^2 B_z^2}{\sinh^2(kd)},$$

where $|h(\omega)|^2$ is now the power-spectrum of the wave heights and the horizontal E-field components vanish.

The above approximate analysis may be done exactly, and with more rigor, by generalizing Weaver’s classical solution for the induced magnetic field generated by linear waves in an infinitely deep sea. This generalization involves the introduction of the sea bottom with both its electromagnetic and hydrodynamic boundary condition and is carried out in the appendix.

In the following we compare measurements and theory. Figure 4 shows the power spectra of the vertical- and horizontal-field components taken at different times and at the relatively shallow depth of 107 meters. The spectrum of $E_z$ has a very clear peak at the dominant wave frequency, while the $E_x$ spectrum is more smeared out. It should be noted that in the course of half a day the strength of the $E_z$ peak decreases by more than an order of magnitude, corresponding to variations in the wave conditions. The same is true for the low frequency part of the $E_z$ spectrum.

In Figure 5 it is seen that the vertical E-field component indeed confirms to the theoretical prediction for angular frequencies below 0.8 Hz, corresponding to wave periods roughly above 10 s. The product of $\sin(\Theta)$ and the magnetic field component $B_z$ is taken to be a reasonable 15 $\mu$T in all three shallow sea figures, as well as in the $d = 313$ m figure. The theory predicts a very sharp decay in the spectrum with frequencies, as the sinh-factor in equation 7 behaves as

$$\sinh^{-2}(kd) \approx e^{-2kd} \approx e^{-2\omega^2 t},$$

in the deep water limit. Frequencies above $\sqrt{\omega/(2d)}$ will therefore cause a quick decay, above which no effect of the waves are

Figure 5. Shallow sea measurements and theoretical predictions of the sea-bottom electric fields at different locations. The black, thick curves show the power spectra of $E_z$, and the magenta curves the associated theory given above. The noisy dashed curve shows the power-spectrum of a horizontal-field component, and the dashed blue curve has a slope of $-2$. The red curves show the ship’s heave power-spectrum multiplied by a squared $B$-field of (15 $\mu$T)$^2$. The theoretical curve is also obtained by setting $B \sin(\Theta) = 15$ $\mu$T. The three different figures correspond to the different depths (a) 107 m, (b) 116 m, and (c) 122 m.
observed. In Figure 6 the cutoff sets in at $\omega = 0.3$ Hz, and in Figure 5 at $\omega = 0.5$ Hz, which is in nice agreement with the above expression. In Figure 5a the dominating wave frequency is a little too high for it to be measured as a peak in the bottom field, while in Figure 5b and 5c the main wave peak is clearly seen in the spectrum. We also see some low frequency oscillations in the $E_x$ spectrum of Figure 5b which are not presently understood.

In all three figures the crossover with increasing $\omega$ from the theoretically predicted wave-signal to some other type of $1/\omega^2$ noise is quite sharp because the wave-signal has such a pronounced decay. Moreover, in Figure 5a and 5b there are significant discrepancies between the low frequency measurements and predictions. This is most likely due to noise in the heave-spectrum. Any kind of random drift in the GPS altitude measurements would contribute a $1/\omega^2$ noise (if the steps in the drift are uncorrelated). The frequencies in the wave spectrum $|h(\omega)|^2$ that lie above the 2 Hz eigenfrequency of the ship are not expected to be detected by the ship motion measurements. However, wave frequencies that lie well below the eigenfrequency are expected to be observable in the GPS measurements. However, because the damping of the hydrodynamic motion with depth effectively removes the effect of the frequencies around the eigenfrequency, this discrepancy between the wave- and ship motion power spectra is mainly of academic interest.

The ratio $E_x/E_v$, which is calculated exactly, for finite $\beta$ in the appendix, is shown in Figure 7, and it is seen that the horizontal component of the wave-induced field is more than two orders of magnitude smaller than the vertical one. This prediction sharply contrasts the measurements, in which the horizontal component exceeds the vertical component by two orders of magnitude outside the dominating frequency of the waves. This means that it must have other causes than the waves and that the wave motion does not dominate as a source in these horizontal E-field measurements. In fact, the measurement of the horizontal component is consistent with the standard magnetotelluric background noise. This may be concluded from independent longer time, deeper sea measurements.

CONCLUSIONS

A general conclusion of our measurements is the electromagnetic noise around the ocean wave frequencies is significantly larger in the horizontal, rather than the vertical, E-field components, and the difference grows with sea depth. At depths around 100 m the power-spectrum of the horizontal components lies 1–2 orders of magnitude above the vertical spectrum, except at the peak corresponding to the maximum of the surface wave spectrum. At the depth of 313 m, the wave effect is screened out, and the difference is 2–3 orders of magnitude over the whole range of frequencies.

The present results do not directly confirm the snapping effect reported by Berdichevsky et al. and Golubev and Zhdanov. This effect of a dominant vertical over horizontal E-field component at the sea bottom is not observed here. Rather, the total power (variance) is much larger in the horizontal component than in the vertical component at large sea depths (300 m), whereas the situation is less clear at shallow depths (120 m). This observation could potentially change in the presence of large 3D contrasts in bathymetry.

The fact that we observe agreement between the field measurements and the prediction of the shallow water wave theory for the vertical but not the horizontal field has several immediate conclusions. First, in the frequency range of the waves, the noise in vertical-field component measurements is indeed dominated by wave-generated field, at least at shallow water. In our $d = 122$ m case the dominant wavelength is roughly 200 m, which is somewhat longer than the depth. In deeper waters it is likely that we will see similar effects when the wavelength-to-depth ratio is the

\begin{verbatim}
Air  \quad \sigma=0
Sea  \quad \sigma_0
Infinite bottom layer \quad \sigma_1
\end{verbatim}

Figure 8. The geologic model.

\begin{verbatim}
Figure 6. Deeper sea measurements and theoretical predictions of the sea bottom electric fields at depth $d = 313$ m. The black, thick curve shows the power-spectrum of the vertical E-field component, and the magenta curve the associated theory given above. The noisy dashed curve shows the power-spectrum of a horizontal-field component, and the stapled blue curve has a slope of $-2$. The red curve shows the ship’s heave power-spectrum multiplied by a squared B-field of 15 μT.

Figure 7. The ratio of the horizontal- to the vertical-field components that are generated by ocean waves at the sea bottom, calculated from equation A-18. The sea depths are $d = 122$ m (full curve) and 300 m (dashed curve).

Figure 8. The geologic model.
\end{verbatim}
same. The conclusion also implies that many noise sources in the vertical channel can be ruled out. For instance, the tidal currents around the receiver station will generally cause Karman streets that produce a hydrodynamic force with a characteristic frequency. Also, small scale turbulence around the receiver station will cause an independent source of noise. Although these effects are likely present, the measurements show that they do not dominate, at least not in the wave-frequency range. Second, the noise in the horizontal-field components must have other causes, likely the standard magnetotelluric ones because the frequency dependence of the spectra are consistent with those. The fact that the vertical predictions and measurements are significantly weaker than the horizontal measurements also means that the vertical signal is not contaminated by the horizontal one. This is interesting, from more than a technological perspective, because it was a priori conceivable that 3D effects of bathymetry could cause the horizontal signal to pollute the vertical one. Third, we see no traces in the E-field measurements of a peak at the double frequency of the main peak of the wave spectrum. Such a peak in the horizontal E-field measurements was pointed out by Longuett-Higgins and was attributed to the standing wave effect described by Longuett-Higgins. The absence of such a peak in our data could be due to an absence of significantly strong wave trains in the opposite direction of the main ones. If so, the effect of standing waves would be weak.

Other noise sources are likely to dominate outside the given frequency range. In the high frequency range, the white electronic noise, with the Nyquist noise as a lower threshold, will dominate. In the low-frequency range electrode drift is one source that is likely to dominate. From a practical point of view, however, the extreme low and extreme high frequency ranges are not problematic when a measurement of a controlled source signal is performed. The reason for this is that the high frequency noise may be averaged away by time window binning, whereas the low-frequency noise may be removed by subtracting signals measured at different times. This is done in some CSEM techniques, in which responses to subsequent signals of opposite polarity are measured. The effect of subtracting responses is to remove any noise components that vary insignificantly over the time between the two measurements. These times must be smaller than the time window of interest for the measurement and the correlation time of the low frequency noise. The frequency range observed in Figure 2 is a problematic range in this context and, therefore, one that is important to understand and control.

Turbulence around the receiver stations will have a velocity field that is not curl-free and, therefore, will provide an independent source of noise that we have not taken into account in the present treatment. However, this noise is likely to have a very different frequency spectrum than that of the surfaces waves.

APPENDIX A

THEORY FOR THE SEA BOTTOM EFFECT OF OCEAN SURFACE WAVES

For the purpose of simplicity, and to make this paper self-contained, we briefly review the 2D theory for the EM fields that are induced by surface gravity waves. A more general treatment, including full 3D fields may be found in Podney (1975), whereas the present treatment agrees with and elaborates on the results of Chave (1983), which in turn generalizes the treatment by Weaver (1965) for an infinitely deep sea.

When the conductive sea water moves due to waves, the Lorentz force created by the earth’s magnetic background field will cause an electric current. This current subsequently sets up a magnetic field that adds to the background field. This magnetic field was calculated previously by Weaver (1965) for an infinitely deep sea and was later generalized to finite sea depths Chave (1983).

Governing equations

When displacement currents are neglected and we use equation 5 and equation 1 as the hydrodynamic starting points, the magnetic field follows from Maxwell’s equations, which may be shown to take the form,

\[
\nabla^2 \mathbf{H} \approx \mu_0 \sigma \frac{\partial \mathbf{H}}{\partial t} - \sigma \mu_0 \mathbf{H}_0 \cdot \nabla \mathbf{u},
\]

where \( \mathbf{H}_0 \) is the background magnetic field, and we have used the fact that the perturbation in the field \( \delta \mathbf{H} = \mathbf{H} - \mathbf{H}_0 \) is much smaller than the background field. Introducing the notation

\[
\delta \mathbf{H} = \mathbf{h}_0 e^{i(\omega + k_z)z} \quad \text{and} \quad \mathbf{u} = \mathbf{v}_0 e^{i(\omega + k_z)z},
\]

the above equation takes the form,

\[
(\partial_x^2 - k^2) \mathbf{h} = i \mu_0 \sigma \omega \mathbf{h} - \mu_0 \sigma \mathbf{H}_0 \cdot \nabla \mathbf{v}.
\]

Now, taking the curl of this equation yields the equation,

\[
(\partial_x^2 - k^2) \mathbf{j} = i \mu_0 \sigma \omega \mathbf{j} - \mu_0 \sigma (\mathbf{H}_0 \cdot \nabla) \times \mathbf{v},
\]

for the current density, where the vorticity \( \nabla \times \mathbf{v} \) shows up as the only source term. A velocity field coming from a plane surface wave moving in the \( x \)-direction is itself constant in the \( y \)-direction with \( u_y = 0 \) automatically giving \( \nabla \times \mathbf{v} = 0 \). Likewise, a velocity field coming from a plane surface wave moving in the \( y \)-direction automatically gives \( \nabla \times \mathbf{v} = 0 \). This means that the vorticity source term always lies in the horizontal plane, and there will be no source to produce a nonvanishing \( j_z \) component. Hence, as long as the sea-earth model is 1D, \( j_z = 0 \) everywhere. However, 3D bathymetry effects will clearly generate a nonvanishing source for \( j_z \).

By the way, it appears paradoxical that there is a current at all because \( \mathbf{u} \) has no \( y \)-component. It may be observed that the last term of equation A-1 with the earth magnetic background field acts as a source term with no \( y \)-component. This means that if the \( \mathbf{H} \)-field is created from zero by gradually turning on the velocity field, \( \mathbf{H}_0 = 0 \) at all times.
Taking the $z$-component of equation A-3 we get
\[
(\partial_z^2 - k^2) h_z = i\mu_0 \sigma \omega h_z - \mu_0 \sigma H_0 \cdot \nabla v_z, \tag{A-5}
\]
where
\[
H_0 \cdot \nabla v_z = -\frac{h_0 k \omega}{\sinh(kd)} (iH_{0z} \sinh(k(z - d)) + H_{0c} \cosh(k(z - d))), \tag{A-6}
\]
so that
\[
(\partial_z^2 - k^2) h_z = i\mu_0 \sigma \omega \left( h_z + \frac{h_0 k}{\sinh(kd)} (H_{0z} \sinh(k(z - d)) - iH_{0c} \cosh(k(z - d))) \right). \tag{A-7}
\]
The homogeneous equation $(\partial_z^2 - k^2) h_z = i\mu_0 \sigma \omega h_z$ has solutions $e^{\pm kv}$, where $q^2 = k^2 + i\mu_0 \sigma \omega$, whereas the particular solution may be found by equating the right-hand side to 0. We will use the notation $q_0$ and $q_1$, where $\sigma = \sigma_0$ and $v = \sigma_1$, respectively. In the air $z < 0$, where $\sigma = 0$, $(\partial_z^2 - k^2) h_z = 0$, and $h_z \propto e^{kv}$. Below the bottom there is no water flow, so $v = 0$, and $h_z$ satisfies the homogeneous equation with $q = q_1$. This means that we can write the solution as
\[
h_z(z) = \frac{h_0 k}{\sinh(kd)} \begin{cases} \frac{Qe^{kv}}{kQ} & \text{when } z < 0 \\
Q \cosh(q_0 d) - P_2 \sinh(q_0 d) + (H_{0z} \sinh(kd) + iH_{0c} \cosh(kd)), \text{ when } d > z > 0, \\
-kQ \cosh(q_0 d) + P_2 q_0 \cosh(q_0 d) - k(H_{0z} \cosh(kd) + iH_{0c} \sinh(kd)), \text{ when } z > d \end{cases} \tag{A-8}
\]
By a bit of straightforward algebra the above equations may be used to verify that, when the conductivity goes to zero, $\sigma \to 0$ and $q_0 \to k$, so there is no current and no induced magnetic field $h_z$. However, this does not imply that there is no electric field in the $\sigma \to 0$ limit, as will be demonstrated below.

Now, to obtain the electric field $E = (1/\sigma) \nabla \times H - \mu_0 \sigma \omega H_0 \cdot \nabla v$, we note that:
\[
\nabla \times H = \frac{ie}{k} (\partial_z^2 - k^2) h_z e^{i(\sigma \omega + kv)}, \tag{A-14}
\]
so that the $\cosh(k(z - d))$ and $\sinh(k(z - d))$ terms above vanish when the curl operator is applied. In the sea layer we are left only with the $e^{-\sigma \omega}$ terms. At the sea bottom $z = d$, the $P_2$ term vanishes, and we get
\[
\nabla \times H = -\frac{ie}{k} \frac{\mu_0 \sigma_0 h_0 \omega}{\sinh(kd)} P_1 e^{i(\sigma \omega + kv)} \tag{A-15}
\]
for the current density $j$ at the sea bottom. The expressions in equation A-15 and equation A-13 give the exact solutions for the magnetic field and current density in an air-sea-earth model. To get the $E$-field we must also calculate,
\[
u \times H \approx u \times H_0 = [-u_x H_y, u_y H_0 - u_z H_0, u_z H_x], \tag{A-16}
\]
which, on the bottom, where $u_x = 0$, becomes
\[
u \times H \approx u \times H_0 = \frac{ih_{0z} \omega}{\sinh(kd)} (e_x H_{0y} - e_y H_{0z}) \tag{A-17}
\]
Using the above in the $E$-field expression gives
\[
E = -\frac{\mu_0 \sigma_0 h_0 \omega}{\sinh(kd)} ((P_1 + iH_{0z}) e_y + iH_{0y} e_x) \tag{A-18}
\]
for the sea bottom $E$-field. Note that the vertical component only contains the Lorentz force contribution and does not contain any effect of the inductive currents. It is the above equation that is used in plotting Figure 7.

In most applications $q_0$ is quite close to $k$. Taylor expands to first order $q_0 \approx k(1 + \hat{r} \hat{t})$, where
\[
\beta = \frac{\mu_0 \sigma_0 \omega}{2k^2} = \frac{\mu_0 \sigma_0 \lambda^2}{4\pi} \approx 0.03. \tag{A-19}
\]
Taking a wavelength of $\lambda = 300$ m and a corresponding frequency from equation 1, $f = 0.1$ Hz, $\beta = 0.003$. Because the $\beta \to 0$ limit is equivalent to the $\sigma_0 \to 0$, or $q_0 \to k$ limit we need the limit,
\[
\lim_{q_0 \to k} P_1 = -iH_{0z}. \tag{A-20}
\]
Combining the above results we obtain
\[
E(d) = e_x \frac{i\mu_0 h_0 \omega}{\sinh(kd)} H_{0z} e^{i(\sigma \omega + kv)}, \tag{A-21}
\]
where the lack of a horizontal-field component comes from the fact that the Lorentz term $\nu \times H$ term cancels the induction term $\nabla \times H$ in this limit.
The infinite-depth limit

It is instructive to compare the above results with the case of an infinitely deep sea, which is described by the $d \to \infty$ limit. In this case the velocity field of the water, which follows from equation 6 is

$$u^\infty(x, t) = h_0(\omega)(e_x - ie_z)e^{i(\omega t - kx - kz)},$$  \hspace{1cm} \text{(A-22)}

where the superscript on $u^\infty$ denotes the $d \to \infty$ limit. Taking the infinite depth limit $d \to \infty$ of equation A-13, it may be shown that

$$h_z(z) \to h_z^\infty(z) = h_0k(H_{0z} + ih_{0w})(e^{j\omega t} - \frac{2k}{k + q_0}e^{-q_0z}).$$  \hspace{1cm} \text{(A-23)}

Note again that when the conductivity goes to zero, $\sigma_0 \to 0$, and $q_0 \to k$, there is no current and no induced magnetic field $h_z$. Using the above velocity field and equation A-23, we get the electric field

$$E^\infty(z) = k\mu_0F_0H(z) + e_{0w}^\infty(H_{0z} + ie_{0w})(e^{i(\omega t - kx - kz)},$$  \hspace{1cm} \text{(A-24)}

which has no $y$-component. This, however, is only true in the $\beta = 0$ limit. If we keep a finite, but small $\beta$-value and Taylor-expand equation A-23 to first order in $\beta$, then the use of equation A-22 gives

$$E^\infty(z) = \beta\mu_0F_0H(z) + ih_{0w}(kz + 1/2)e^{j\omega t - kx - kz},$$  \hspace{1cm} \text{(A-25)}

as obtained by Cox et al. (1978). The smallness of $\beta$, though, will make $E^\infty \approx E^r$, or $E^\infty$. To compare the infinite sea depth- and the bottom values of the fields, we take the large $kd$ limit of equation A-21, which, by the approximation $\sinh(kd) \approx e^{kd}/2$, becomes

$$E(d) = e_{0w}^2 \mu_0h_0\omega H_0e^{i(\omega t + kx - k)d},$$  \hspace{1cm} \text{(A-26)}

which, apart from not having an $x$-component, has twice the magnitude of the infinite depth field $E^\infty(d)$. The fact that the bottom field has no $x$-component simply reflects the fact that the water does not flow through the sea floor, so that the bottom velocity field has no $z$-component. The doubling of the magnitude of the bottom field, compared to the deep-water field at the same depth $z = d$, is attributed to an increase in the horizontal water velocity due to the presence of the bottom. It is easily seen from the large $kd$ limit of equation 6 that the bottom velocity $u_z(z = d)$ is twice the velocity $u^\infty(z = d)$ calculated from equation A-22.

Power spectra

Now, if $h_0$ is taken to be only of many frequency components in the ocean-wave spectrum $h(\omega)$, we can use that fact that all the above results are linear in $h_0$ to generalize the above expressions to hold for an arbitrary wave spectrum $h(\omega)$. This gives the following expressions for the bottom sea power-spectrum of the vertical-field component:

$$|E(d, \omega)|^2 \approx |E^r(d, \omega)|^2 = \frac{|h(\omega)|^2\sigma B^2_z}{\sinh^2(kd)}.$$  \hspace{1cm} \text{(A-27)}

where we have used $B = \mu_0H_0$, and where the frequency and wave number is related through the dispersion relation of equation 1. The approximation $|E(d, \omega)|^2 \approx |E^r(d, \omega)|^2$ holds to order $\beta \sim 10^{-3}$, as is shown above.

REFERENCES